JOURNAL OF FUNCTIONAL ANALYSIS 33, 231–258 (1979)

A Theory of Numerical Range for Nonlinear Operators

JOSÉ A. CANAVATI

IIMAS, UNAM, Apartado Postal 20-726, México 20, D.F., México

Communicated by A. P. Calderon

Received July 2, 1976; revised February 15, 1978

1. INTRODUCTION

Let H be a Hilbert space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), with inner product $(\cdot \mid \cdot)$. For a bounded linear operator T on H, the numerical range has a very natural definition which was introduced, in the finite dimensional case, by Toeplitz in 1918 [9] as follows: The numerical range $W(T)$ of T is the set of scalars defined by

$$W(T) = \{(Tx \mid x);\ \|x\| = 1\}. \tag{1}$$

The numerical radius $w(T)$ of T is defined as the number

$$w(T) = \sup\{|\lambda|;\ \lambda \in W(T)\}. \tag{2}$$

The following are some known facts in the linear theory [2], [3], [6].

(a) The numerical range of an operator is always convex.

(b) The closure of the numerical range of an operator contains its spectrum.

(c) If $\mu \in G$ is such that $\delta = \operatorname{dist}(\mu, W(T)) > 0$, then

$$\|(\mu I - T)^{-1}\| \leqslant \delta^{-1}.$$

No concept of numerical range appropriate to general Banach spaces appeared until 1961 and 1962, when distinct though related, concepts were introduced independently by Lumer [8] and Bauer [1]. Lumer defined the concept of semi-inner product on a Banach space as follows. We say that a complex (real) semi-inner product is defined on a complex (real) Banach space X, if to any x, y in X there corresponds a complex (real) number $[x, y]$ and the following properties hold:

(i) $[x + y, z] = [x, z] + [y, z]$ for x, y, z in X.

(ii) $[\lambda x, y] = \lambda[x, y]$ for x, y in X, λ complex (real).

0022-1236/79/090231–28$02.00/0

(iii) $[x, x] > 0$ for $x \neq 0$.

(iv) $|[x, y]|^2 \leqslant [x, x][y, y]$ for x, y in X.

Lumer showed that every Banach space $(X, \Box \cdot \Box)$ has at least one semi-inner product $[\cdot, \cdot]$ such that

(v) $[x, x] = \Box x \Box^2$, $x \in X$.

In terms of a semi-inner product, the definition of numerical range used for Hilbert spaces at once generalizes to give the definition of the numerical range $W(T)$ for bounded linear operators T on X, as follows

$$W(T) = \{[Tx, x];\ \Box x \Box = 1\}. \tag{3}$$

This definition has the serious defect that is not an invariant of the Banach space $(X, \Box \cdot \Box)$, since, except when the unit ball of X is smooth, there are infinitely many semi-inner products on X satisfying (v). However, this defect is more apparent than real, for Lumer proved the formula

$$\sup\{\operatorname{Re}\lambda;\ \lambda \in W(T)\} = \lim_{\alpha \to 0+} \frac{\Box I + \alpha T \Box - 1}{\alpha}, \tag{4}$$

from which follows that $\overline{co}\,W(T)$, the closed convex hull of $W(T)$, is independent of the choice of semi-inner product satisfying (v).

Bauer's paper [1] was concerned only with finite dimensional Banach spaces, but the concept of numerical range that he introduced is available without restriction of the dimension. Let X be a Banach space over $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), X^* its dual space, and denote by $\langle x, x^* \rangle$ ($x \in X$, $x^* \in X^*$) the duality map between X and X^*. Then for any bounded linear operator T on X, the "spatial" numerical range $V(T)$ is defined by

$$V(T) = \{\langle Tx, x^* \rangle;\ \Box x \Box = \Box x^* \Box = \langle x, x^* \rangle = 1\}, \tag{5}$$

and the spatial numerical radius $v(T)$ of T is the number

$$v(T) = \sup\{|\lambda|;\ \lambda \in V(T)\}. \tag{6}$$

A result by Bonsall, Cain and Schneider [2] assures that $V(T)$ is a connected subset of $\mathbb{K}$, unless X has dimension one over $\mathbb{R}$; and Crabb [3] has shown that the convex hull of the spectrum of T is contained in the closure $\overline{V(T)}$ of the spatial numerical range of T.

When X is a Hilbert space, $V(T)$ coincides with the classical numerical range. If X is a Banach space with a smooth unit ball, then $V(T)$ coincides with the numerical range $W(T)$ corresponding to the unique semi-inner product $[\cdot, \cdot]$ satisfying (v). For a general Banach space X, $V(T)$ is the union of all the numerical ranges $W(T)$, corresponding to all choices of semi-inner products satisfying (v); and for each choice of semi-inner product,

$$\overline{co}V(T) = \overline{co}W(T). \tag{7}$$

The proofs of these essertions can be found in [2] and [3].

In [4] Furi and Vignoli defined a numerical range for the class of all quasibounded (nonlinear) maps on a Hilbert space H, and gave some of the basic properties of such numerical range. What we shall do here is to define a numerical range for a broader class of maps; the "numerically bounded" maps on a Banach space X, and study it in a more systematic way. Among other properties, our numerical range will be compact and connected, and will coincide with $\overline{V(T)}$, in the particular case when T is a bounded linear operator on a Banach space X. That our numerical range is already closed does not pose any particular problem since, as far spectral theory is concerned, it is the closure of the numerical range (e.g., Crabb's result) the set who plays an important role.

The plan of the work is as follows: In Section 2 we will define some Banach spaces of nonlinear maps which are going to be the object of study in this work. For reasons that are going to be apparent in later sections, we found more convenient to deal with maps of the form F: $X \times X^* \to X$, instead of maps f: $X \to X$, the later being a particular case of the former. In Section 3 we will define the *-numerical range for the maps F; and in Section 4 we will see what form take our previous results when dealing with the maps f. In Section 5 we are going to define the *-asymptotic spectrum of a map F and study its relations with the *-numerical range. Section 6 is devoted to the purpose of obtaining ana analoge of Lumer's formula (4) for the class of Lipschitz maps. In Section 7 we will introduce the concept of (nonlinear) adjoint of a numerically bounded map, and we will show that they have the same numerical range. In Section 8, as in [5], we will obtain some surjectivity results for compact numerically bounded maps. Finally, in Section 9 we are going to define the numerical range for the numerically bounded vector fields on the unit sphere of a Banach space.

To end this section we are going to review some elementary definitions.

DEFINITION 1.1. Let X be a Banach space over $\mathbb{K}$.

(a) $L(X)$ denotes the Banach space of all bounded linear operators on X.

(b) $B(X)$ is the vector space of all continuous maps f: $X \to X$ such that $\square f(x) \square \leqslant M \square x \square$, for some $M \geqslant 0$ and all x in X. We define the norm $\square f \square$ of f as the smallest $M \geqslant 0$ such that this inequality holds for all x in X. An element of $B(X)$ is called a bounded map on X. It is clear that $B(X)$ is a Banach space.

(c) $Q(X)$ is the vector space of all quasibounded maps on X. That is, the space of all continuous maps $f: X \to X$ such that there exists $A, B \geqslant 0$ satisfying

$$\|f(x)\| \leqslant A + B\|x\|, \; x \in X. \tag{1}$$

Define $|f|$, the quasinorm of f, to be the infimum of all $B \geqslant 0$ for which (1) holds for some $A \geqslant 0$, i.e.,

$$|f| = \limsup_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|}.$$

Notice that $|\cdot|$ is a seminorm on $Q(X)$.

(d) Let $f, g \in Q(X)$. The mapping f is said to be "asymptotically equivalent" to g if $|f - g| = 0$. It is easy to see that this is an equivalence relation. If f is asymptotically equivalent to a $T \in L(X)$, then we say that f is an "asymptotically linear" map.

(e) $\tilde{Q}(X)$ is the normed space of all equivalence classes of quasibounded maps, i.e., $\tilde{Q}(X) = Q(X)/N(|\cdot|)$, where $f \in N(|\cdot|)$ iff $|f| = 0$. The norm in $\tilde{Q}(X)$ is the one induced by $|\cdot|$, and will be denoted in the same way. It is shown in [5] that $\tilde{Q}(X)$ is a Banach space.

2. Some Banach Spaces of Nonlinear Maps

Definition 2.1. The norm $\times$ weak* topology in $X \times X^*$, is the topology in $X \times X^*$ given by the norm topology on X and the weak* topology on X^*.

We define the following subsets of $X \times X^*$.

$$\Pi_r = \{(x, x^*) \in X \times X^*; \; \|x\| = \|x^*\| \geqslant r, \; \|x\|^2 = \langle x, x^* \rangle\},$$

for $r > 0$, and

$$\Pi_0 = \bigcup_{r>0} \Pi_r.$$

The following two results are essentially due to Bonsall, Cain and Schneider [2; 100–103].

Lemma 2.1. *Let π denote the natural projection of $X \times X^*$ onto X, and let A be a subset of Π_r that is relatively closed in Π_r with respect to the norm $\times$ weak* topology. Then $\pi(A)$ is a (norm) closed subset of X.*

Proof. Let $\{x_n\}$ be a sequence in $\pi(A)$ such that $x_n \to x_0 \in X$. There exists a sequence $\{x_n^*\}$ in X^* such that $\{(x_n, x_n^*)\} \subseteq A$. In particular, the sequence $\{x_n^*\}$ is bounded in X^* (since $\{x_n\}$ is convergent and $\|x_n\| = \|x_n^*\|$). By the weak*-compactness of the closed unit ball in X^*, there exists an $x_0^* \in X^*$ such that

$x_n^* \rightharpoonup x_0^*$, and $\|x_0^*\| \leqslant \lim\ \inf\|x_n^*\| = \lim\|x_n\| = \|x_0\|$. Where the corresponding subsequence of $\{x_n^*\}$ has been denoted by the same symbol $\{x_n^*\}$. We have

$$\langle x_0 , x_0^*\rangle = \langle x_0 , x_0^* - x_n^*\rangle + \langle x_0 - x_n , x_n^*\rangle + \langle x_n , x_n^*\rangle,$$

and so

$$|\langle x_0 , x_0^*\rangle - \|x_0\|^2| \leqslant |\langle x_0 , x_0^* - x_n^*\rangle| + \|x_n^*\|\,\|x_0 - x_n\| + |\|x_n\|^2 - \|x_0\|^2|.$$

Since the right hand side of this last inequality tends to zero as n goes to infinity, we obtain $\langle x_0 , x_0^*\rangle = \|x_0\|^2$; and hence $\|x_0\| \leqslant \|x_0^*\|$. Thus we have shown that $\|x_0\| = \|x_0^*\| \geqslant r$, $\langle x_0 , x_0^*\rangle = \|x_0\|^2$, i.e., $(x_0 , x_0^*) \in A$ and $\pi(x_0 , x_0^*) = x_0 \in \pi(A)$. Therefore $\pi(A)$ is closed.

Proposition 2.1. *Each Π_r $(r > 0)$ and Π_0 are connected subsets of $X \times X^*$ with the norm $\times$ weak* topology, unless X has dimension one over $\mathbb{R}$.*

Proof. First we show that each Π_r is connected. Suppose we have $\Pi_r = A \cup B$, where A, B are relatively closed in Π_r and $A \cap B = \varnothing$. The previous lemma implies that $\pi(A)$ and $\pi(B)$ are norm closed subsets of X, and $\pi(A) \cup \pi(B) = \{x \in X;\ \|x\| \geqslant r\}$. Suppose that $x_0 \in \pi(A) \cap \pi(B)$. Then there exists x_1^*, $x_2^* \in X^*$ such that $(x_0 , x_1^*) \in A$ and $(x_0 , x_2^*) \in B$. We have for $0 \leqslant t \leqslant 1$ $\langle x_0 , tx_1^* + (1 - t_2)\, x^*\rangle = t\langle x_0 , x_1^*\rangle + (1 - t)\langle x_0 , x_2^*\rangle = \|x_0\|^2$, and hence $\|tx_1^* + (1 - t)\, x_2^*\| \geqslant \|x_0\|$ $(0 \leqslant t \leqslant 1)$. Also $\|tx_1^* + (1 - t)\, x_2^*\| \leqslant t\|x_1^*\| + (1 - t)\|x_2^*\| = \|x_0\|$. Thus we have shown that

$$(x_0 , tx_1^* + (1 - t)\, x_2^*) \in \Pi_r \qquad (0 \leqslant t \leqslant 1),$$

which is impossible since $A \cap B = \varnothing$. Therefore $\pi(A) \cap \pi(B) = \varnothing$. Now, if X does not have dimension one over $\mathbb{R}$, then the set $\{x;\ \|x\| \geqslant r\}$ is connected. Thus we must have $\pi(A) = \varnothing$ or $\pi(B) = \varnothing$. Therefore Π_r is connected. That Π_0 is connected follows immediately from [7; 228], since $\{\Pi_r;\ r > 0\}$ is a directed family of connected subsets of $X \times X^*$.

From now on we shall assume that Π_0 has the norm $\times$ weak* topology induced as a subset of $X \times X^*$. Also we shall assume that X does not have dimension one over $\mathbb{R}$. Our theory being trivial in this case.

Definition 2.2. Let $F\colon \Pi_0 \to X$ be a continuous map. We say that F is "*-bounded" if

$$\|F\|_* = \sup_{\Pi_0} \frac{\|F(x, x^*)\|}{\|x\|} < \infty.$$

We denote by $B_*(X)$, the vector space of all *-bounded maps. Notice that $\|\cdot\|_*$ is a norm on $B_*(X)$.

We can consider the vector space $B(X)$ as a vector subspace of $B_*(X)$ in a natural way, namely; if $f \in B(X)$, then the mapping $F(x, x^*) = f(x)$ belongs to $B_*(X)$ and $\|f\| = \|F\|_*$.

PROPOSITION 2.2. *$B_*(X)$ is a Banach space.*

Proof. This is a standard argument, and so it will be omitted.

DEFINITION 2.3. Let $F\colon \Pi_0 \to X$ be a continous map. We say that F is "*-quasibounded" if

$$|F|_* = \lim_{r\to\infty} \sup_{\Pi_r} \frac{\|F(x, x^*)\|}{\|x\|} < \infty.$$

We denote by $Q_*(X)$ the vector space of all *-quasibounded maps. Notice that $|\cdot|_*$ is a seminorm on $Q_*(X)$. Obviously one has $B_*(X) \subset Q_*(X)$ and $|F|_* \leqslant \|F\|_*$. By elementary examples it is easily seen that the inclusion is proper.

We can consider the vector space $Q(X)$ as a vector subspace of $Q_*(X)$ in a natural way, namely; if $f \in Q(X)$, then the mapping $F(x, x^*) = f(x)$, belongs to $Q_*(X)$ and $|f| = |F|_*$.

PROPOSITION 2.3. *For any $F \in Q_*(X)$, there exists a sequence $\{F_n\}$ in $B_*(X)$ such that $|F_n - F|_* = 0$ $(n = 1, 2, 3,...)$ and $\|F_n\|_* \to |F|_*$ as $n \to \infty$*

Proof. Let $\rho^2 = \|x\|^2 + \|x^*\|^2$, and define $F_n(x, x^*) = F(x, x^*)$ if $\rho \geqslant n$, $F_n(x, x^*) = \rho/nF(n/\rho x, n/\rho x^*)$ if $0 < \rho < n$.

We have

$$\|F_n\|_* = \sup_{\Pi_0} \frac{\|F_n(x, x^*)\|}{\|x\|} = \sup_{\Pi_{n/\sqrt{2}}} \frac{\|F(x, x^*)\|}{\|x\|}.$$

Therefore $F_n \in B_*(X)$ for all n large enough and $\|F_n\|_* \to |F|_*$ as $n \to \infty$.

DEFINITION 2.4. (a) Let $F, G \in Q_*(X)$. The mapping F is said be "*-asymptotically equivalent" to G (*-a.e.) if $|F - G|_* = 0$. It is easy to see that this is an equivalence relation. (b) $\tilde{Q}_*(X)$ is the normed space of all equivalence classes of *-quasibounded maps, i.e., $\tilde{Q}_*(X) = Q_*(X)/N(|\cdot|_*)$, where $F \in N(|\cdot|_*)$ iff $|F|_* = 0$. The norm on $\tilde{Q}_*(X)$ is the induced by $|\cdot|_*$ and will be denoted in the same way.

From Proposition 2.3 we see that the mapping

$$B_*(X) \to \tilde{Q}_*(X), \quad F \to \tilde{F}$$

is onto. Furthermore we have:

PROPOSITION 2.4. *$\tilde{Q}_*(X)$ is a Banach space.*

Proof. Let $\{\tilde{F}_n\}$ be a sequence in $\tilde{Q}_*(X)$ such that $\sum |\tilde{F}_n|_* < \infty$. We have to show that $\sum \tilde{F}_n$ converges. By Proposition 2.3, for any positive integer n we can choose $G_n \in B_*(X)$ such that $\tilde{G}_n = \tilde{F}_n$ and $\|G_n\|_* \leqslant |F_n|_* + 2^{-n}$. Since $B_*(X)$ is Banach, $\sum G_n$ converges to an element $G \in B_*(X)$. From the continuity of the linear projection $B_*(X) \to \tilde{Q}_*(X)$ we obtain $\sum \tilde{G}_n = \sum \tilde{F}_n = \tilde{G}$. Which is what we wanted to prove.

DEFINITION 2.5. Let $F: \Pi_0 \to X$ be a continous map. We say that F is "*-numerically bounded" if

$$\omega_*(F) = \lim_{r\to\infty} \sup_{\Pi_r} \frac{|\langle F(x, x^*), x^*\rangle|}{\|x\| \, \|x^*\|} < \infty.$$

We denote by $W_*(X)$ the vector space of all *-numerically bounded maps. Notice that ω_* is a seminorm on $W_*(X)$. If $F \in W_*(X)$, then we let

$$\alpha_*(F) = \lim_{r\to\infty} \inf_{\Pi_r} \frac{|\langle F(x, x^*), x^*\rangle|}{\|x\| \, \|x^*\|}.$$

Obviously one has $Q_*(X) \subset W_*(X)$ and $\omega_*(F) \leqslant |F|_*$. By elementary examples (see Section 4) one can see that the inclusion is proper.

DEFINITION 2.6. Let $F \in W_*(X)$ and consider the maps

$$F_\nu: \Pi_0 \to X \qquad \text{and} \qquad F_\tau: \Pi_0 \to X,$$

given by

$$F_\nu(x, x^*) = \frac{\langle F(x, x^*), x^*\rangle}{\|x\| \, \|x^*\|}\, x$$

and

$$F_\tau(x, x^*) = F(x, x^*) - F_\nu(x, x^*).$$

Then $F = F_\nu + F_\tau$. The maps F_ν and F_τ are called the "normal" and "tangent" components of F respectively.

LEMMA 2.2. *Let $F \in W_*(X)$. Then:*

(a) $\langle F_\nu(x, x^*), x^*\rangle = \langle F(x, x^*), x^*\rangle$, $(x, x^*) \in \Pi_0$.

(b) $\langle F_\tau(x, x^*), x^*\rangle = 0$, $(x, x^*) \in \Pi_0$.

(c) $F_\nu \in Q_*(X)$ *and* $|F_\nu|_* = \omega_*(F)$.

Proof. They follow immediately from the definitions.
The following result is also obvious.

PROPOSITION 2.5. *Let $F\colon \Pi_0 \to X$ be a continuous map. Then $F \in W_*(X)$ if and only if there exists continuous mappings $G, H\colon \Pi_0 \to X$ with $G \in Q_*(X)$ and H satisfying*

$$\langle H(x, x^*), x^* \rangle = 0 \ ((x, x^*) \in \Pi_0),$$

such that $F = G + H$. Such a map H is said to be an $*$-orthogonal map.

DEFINITION 2.7. (a) Let $F, G \in W_*(X)$. The mapping F is said to be '$*$-asymptotically numerically equivalent'' ($*$-a.n.e.) to G if $\omega_*(F - G) = 0$. It is easy to see that this is an equivalence relation.

(b) $\hat{W}_*(X)$ is the normed space of all equivalence classes of $*$-numerically bounded maps, i.e., $\hat{W}_*(X) = W_*(X)/N(\omega_*)$, where $F \in N(\omega_*)$ iff $\omega_*(F) = 0$. The norm on $\hat{W}_*(X)$ is the one induced by ω_*, and it will be denoted in the same way.

Now, let

$$\tilde{}\colon Q_*(X) \to \tilde{Q}_*(X) \qquad \text{and} \qquad \hat{}\colon W_*(X) \to \hat{W}_*(X)$$

be the natural linear projections. Then we have the following commutative diagram of continuous linear maps

$$\begin{tikzcd}
W_*(X) \arrow[r, "\wedge"] & \hat{W}_*(X) \\
Q_*(X) \arrow[u, "j"] \arrow[ur, "q"] \arrow[r, "\sim"] & \tilde{Q}_*(X) \arrow[u, "r"']
\end{tikzcd}$$

where j is the inclusion map of $Q_*(X)$ into $W_*(X)$, $q(F) = \hat{F}$ and $r(\tilde{F}) = \hat{F}$. Note that the map r is well defined, because if $F, G \in Q_*(X)$ are such that $\tilde{F} = \tilde{G}$, then $\omega_*(F - G) \leqslant |F - G|_* = 0$, and hence $\hat{F} = \hat{G}$.

PROPOSITION 2.6. *$\hat{W}_*(X)$ is a Banach space.*

Proof. Let $\{\hat{F}_n\}$ be a sequence in $\hat{W}_*(X)$ such that $\sum \omega_*(\hat{F}_n) < \infty$. We have to show that $\sum \hat{F}_n$ converges. Since

$$\omega_*(\hat{F}) = \omega_*(F) = |F_\nu|_* = |\tilde{F}_\nu|_* \qquad (F \in W_*(X)),$$

where $F_\nu \in Q_*(X)$ (Lemma 2.2) is the normal component of F, then we have

$$\sum |\tilde{F}_{n\nu}|_* = \sum \omega_*(\hat{F}_n) < \infty. \tag{1}$$

But $\{\tilde{F}_{n\nu}\}$ is a sequence in the Banach space $\tilde{Q}_*(X)$, and it follows from (1) and

Proposition 2.5 that the series $\sum \tilde{F}_{n\nu}]$ converges to an element $\tilde{F} \in \tilde{Q}_*(X)$. Since the mapping $r: \tilde{Q}_*(X) \to \hat{W}_*(X)$ is linear and continuous we must have

$$\sum \hat{F}_{n\nu} = \sum r(\tilde{F}_{n\nu}) = r(\tilde{F}) = \hat{F}. \tag{2}$$

But $\hat{F} = \hat{F}_\nu$ for $F \in W_*(X)$. Hence from (2) we obtain $\sum \hat{F}_n = \hat{F}$. Which is precisely what we wanted to show.

3. The *-Numerical Range

Definition 3.1. Let $F \in W_*(X)$ and consider the continuous map $\phi_F: \Pi_0 \to \mathbb{K}$, given by

$$\phi_F(x, x^*) = \frac{\langle F(x, x^*), x^* \rangle}{\|x\| \, \|x^*\|}.$$

We define the *-numerical range $\Omega_*(F)$ of F as the set

$$\Omega_*(F) = \bigcap_{r>0} \overline{\phi_F(\Pi_r)}.$$

In other words, $\lambda \in \Omega_*(F)$ if and only if there exists a sequence $\{(x_n, x_n^*)\}$ in Π_0 such that $\|x_n\| \geqslant n$ and

$$\frac{\langle F(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} \to \lambda \qquad \text{as} \quad n \to \infty.$$

Proposition 3.1. *If $F \in W_*(X)$, then $\Omega_*(F)$ is a nonempty compact connected subset of $\mathbb{K}$.*

Proof. Since $F \in W_*(X)$, then the sets $\overline{\phi_F(\Pi_r)}$ are bounded for all $r > 0$ large enough. Now $\{\overline{\phi_F(\Pi_r)}\}$ is a nested family of compact nonempty sets, therefore by Cantor's theorem $\Omega_*(F) \neq \varnothing$ and is compact. Now from Proposition 2.1 we have that each $\overline{\phi_F(\Pi_r)}$ is a connected subset of $\mathbb{K}$. Thus $\Omega_*(F)$ being an intersection of a nested family of compact connected sets is connected as well [7; 236].

The following properties of the *-numerical range are easy to check.

Proposition 3.2. *Let F, $G \in W_*(X)$ and $\mu \in \mathbb{K}$. Then:*

(a) $\Omega_*(F_\nu) = \Omega_*(F)$ *and* $\Omega_*(F_\tau) = \{0\}$.

(b) $\Omega_*(\mu F) = \mu \Omega_*(F)$.

(c) $\Omega_*(\mu\pi + F) = \mu + \Omega_*(F)$, *where* $\pi: X \times X^* \to X$ *denotes the natural projection.*

(d) $\Omega_*(F + G) \subseteq \Omega_*(F) + \Omega_*(G)$.

PROPOSITION 3.3. *If* $F \in W_*(X)$, *then*

$$\omega_*(F) = \max\{| \lambda |;\ \lambda \in \Omega_*(F)\}.$$

We call $\omega_*(F)$ *the "*-numerical radius" of* F.

Proof. It follows immediately from the definitions of $\Omega_*(F)$ and $\omega_*(F)$.

PROPOSITION 3.4. *If* F, $G \in W_*(X)$ *and* $\omega_*(F - G) = 0$, *then* $\Omega_*(F) = \Omega_*(G)$.

Proof. From Proposition 3.2 we have $\Omega_*(F) = \Omega_*(F_\nu)$ and $\Omega_*(G) = \Omega_*(G_\nu)$. Also from Lemma 2.2(c) we have $| F_\nu - G_\nu |_* = \omega_*(F - G) = 0$. We shall show that $\Omega_*(F_\nu) = \Omega_*(G_\nu)$. Let $\lambda \in \Omega_*(F_\nu)$, then there is a sequence $\{(x_n, x_n^*)\}$ in Π_0 such that $\|x_n\| \geqslant n$ and

$$\frac{\langle F(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} \to \lambda \qquad \text{as} \quad n \to \infty.$$

Now

$$\frac{\langle G_\nu(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} = \frac{\langle (G_\nu - F_\nu)(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} + \frac{\langle F_\nu(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} \tag{1}$$

But

$$\frac{|\langle (G_\nu - F_\nu)(x_n, x_n^*), x_n^* \rangle|}{\|x_n\| \, \|x_n^*\|} \leqslant \frac{\|(G_\nu - F_\nu)(x_n, x_n^*)\|}{\|x_n\|},$$

and $| G_\nu - F_\nu |_* = 0$ imply

$$\frac{\langle (G_\nu - F_\nu)(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} \to 0. \tag{2}$$

Hence from (1) and (2) we see that

$$\frac{\langle G_\nu(x_n, x_n^*), x_n^* \rangle}{\|x_n\| \, \|x_n^*\|} \to \lambda.$$

Therefore $\Omega_*(F_\nu) \subseteq \Omega_*(G_\nu)$. The inclusion $\Omega_*(G_\nu) \subseteq \Omega_*(F_\nu)$ is proved in the same way.

PROPOSITION 3.5. *If* $F \in W_*(X)$, *then*

$$\alpha_*(\mu\pi - F) \geqslant \operatorname{dist}(\mu, \Omega_*(F)),\ \mu \in \mathbb{K}.$$

Proof. We shall show a little more, namely; that for any $\mu \in \mathbb{K}$, there exists $\lambda \in \Omega_*(F)$ such that $\alpha_*(\mu\pi - F) = |\mu - \lambda|$. By definition of $\alpha_*(\mu\pi - F)$, there is a sequence $\{(x_n, x_n^*)\}$ in Π_0 such that $\square x_n \square \geqslant n$ and

$$\frac{|\langle(\mu\pi - F)(x_n, x_n^*), x_n^*\rangle|}{\square x_n \square\ \square x_n^* \square} \to \alpha_*(\mu\pi - F). \tag{1}$$

Since $F \in W_*(X)$, without loss of generality we may assume that the sequence

$$\left\{\frac{\langle F(x_n, x_n^*), x_n^*\rangle}{\square x_n \square\ \square x_n^* \square}\right\}$$

is convergent to some $\lambda \in \Omega_*(F)$. Thus from (1) we obtain $\alpha_*(\mu\pi - F) = |\mu - \lambda|$.

Next we give another characterization of the *-numerical range.

PROPOSITION 3.6. *If* $F \in W_*(X)$, *then*

$$\Omega_*(F) = \{\lambda \in \mathbb{K};\ \alpha_*(\lambda\pi - F) = 0\}.$$

Proof. Let $\Lambda = \{\lambda \in \mathbb{K};\ \alpha_*(\lambda\pi - F) = 0\}$. Then from the previous proposition we have $\Lambda \subseteq \Omega_*(F)$. Now let $\lambda \in \Omega_*(F)$. Then there is a sequence $\{(x_n, x_n^*)\}$ in Π_0 such that $\square x_n \square \geqslant n$ and

$$\frac{\langle F(x_n, x_n^*), x_n^*\rangle}{\square x_n \square\ \square x_n^* \square} \to \lambda.$$

This, in turn, implies that

$$\frac{\langle(\lambda\pi - F)(x_n, x_n^*), x_n^*\rangle}{\square x_n \square\ \square x_n^* \square} \to 0;$$

and hence that $\alpha_*(\lambda\pi - F) = 0$. Therefore $\lambda \in \Lambda$ and $\Omega_*(F) \subseteq \Lambda$.

PROPOSITION 3.7. *Let* $F, G \in W_*(X)$ *and* $\mu \in \mathbb{K}$. *Then*:

(a) $0 \leqslant \alpha_*(F) \leqslant \omega_*(F)$.

(b) $\alpha_*(\mu F) = |\mu|\,\alpha_*(F)$.

(c) $\alpha_*(F + G) \leqslant \alpha_*(F) + \omega_*(G)$.

(d) $\alpha_*(F) - \omega_*(G) \leqslant \alpha_*(F + G)$.

(e) $|\alpha_*(F) - \alpha_*(G)| \leqslant \omega_*(F - G)$. *So* α_* *is actually defined in* $\hat{W}_*(X)$.

(f) $\alpha_*(F) \leqslant |\lambda|$ *if* $\lambda \in \Omega_*(F)$.

Proof. (a) and (b) follow from the definitions.

(c) $$\alpha_*(F+G) = \lim_{r\to\infty}\inf_{\Pi_r} \frac{|\langle (F+G)(x,x^*), x^*\rangle|}{\|x\|\,\|x^*\|}$$
$$\leqslant \lim_{r\to\infty}\inf_{\Pi_r} \frac{|\langle F(x,x^*), x^*\rangle|}{\|x\|\,\|x^*\|} + \lim_{r\to\infty}\sup_{\Pi_r} \frac{|\langle G(x,x^*), x^*\rangle|}{\|x\|\,\|x^*\|}$$
$$\leqslant \alpha_*(F) + \omega_*(G).$$

(d) It follows from (c).

(e) We have from (c)

$$\alpha_*(F) = \alpha_*(F - G + G) \leqslant \omega_*(F-G) + \alpha_*(G).$$

Hence $|\alpha_*(F) - \alpha_*(G)| \leqslant \omega_*(F-G)$.

(f) From (d) and Proposition 3.6 we have

$$\alpha_*(F) - |\lambda| \leqslant \alpha_*(\lambda\pi - F) = 0, \ \lambda \in \Omega_*(F).$$

Recall that if (M, d) is a metric space and $\Gamma(M)$ denotes the set of all non-void closed bounded subsets of M, and if we define

$$\gamma(A, B) = \max\{\sup_{x\in B} \operatorname{dist}(x, A), \sup_{x\in A} \operatorname{dist}(x, B)\}, \qquad A, B \in \Gamma(M).$$

Then $(\Gamma(M), \gamma)$ is a metric space. The metric γ is called the Hausdorff metric.

PROPOSITION 3.8. *If $F, G \in W_*(X)$, then*

$$\gamma(\Omega_*(F), \Omega_*(G)) \leqslant \omega_*(F-G). \tag{1}$$

Here γ is the Hausdorff metric in $\Gamma(\mathbb{K})$.

Proof. We have

$$\gamma(\Omega_*(F), \Omega_*(G)) = \max\{\sup\{\operatorname{dist}(\lambda, \Omega_*(F)); \lambda \in \Omega_*(G)\},$$
$$\sup (\operatorname{dist}(\lambda, \Omega_*(G)); \lambda \in \Omega_*(F)\}\}, \tag{2}$$

and from Proposition 3.5

$$\operatorname{dist}(\lambda, \Omega_*(F)) \leqslant \alpha_*(\lambda\pi - F), \ \operatorname{dist}(\lambda, \Omega_*(G)) \leqslant \alpha_*(\lambda\pi - G). \tag{3}$$

Also propositions 3.6 and 3.7(c) imply

$$\alpha_*(\lambda\pi - F) = \alpha((\lambda\pi - G) + (G - F))$$
$$\leqslant \alpha_*(\lambda\pi - G) + \omega_*(G - F)$$
$$= \omega_*(F - G), \ \lambda \in \Omega_*(G), \tag{4}$$

and

$$\begin{aligned}\alpha_*(\lambda\pi - G) &= \alpha_*((\lambda\pi - F) + (F - G)) \\ &\leqslant \alpha_*(\lambda\pi - F) + \omega_*(F - G) \\ &= \omega_*(F - G),\ \lambda \in \Omega_*(F).\end{aligned}$$

From (2)–(5) we obtain (1).

4. The Numerical Range

Definition 4.1. Let $X_0 = X - \{0\}$, and $f\colon X_0 \to X$ be a continuous map. We say that f is a "numerically bounded" map, if the map $F\colon X_0 \to X$ given by $F(x, x^*) = f(x)$ is *-numerically bounded, i.e.,

$$\lim_{r\to\infty} \sup_{\Pi_r} \frac{|\langle f(x), x^*\rangle|}{\|x\|\ \|x^*\|} < \infty.$$

In this case the numbers $\omega_*(F)$, $\alpha_*(F)$ and the *-numerical range $\Omega_*(F)$ are denoted by $\omega(f)$, $\alpha(f)$ and $\Omega(f)$ respectively. We denote by $W(X)$ the vector space consisting of all numerically bounded maps on X_0. Notice that $W(X)$ can be considered, in a natural way, as a vector subspace of $W_*(X)$, and that ω is a seminorm on $W(X)$. Obviously one has $B(X) \subset Q(X) \subset W(X)$ and $\omega(f) \leqslant |f| \leqslant \|f\|$.

That the inclusion $Q(X) \subset W(X)$ is proper is shown in the following example: Let $X = \mathbb{R}^2$ and $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be given by $f(x_1, x_2) = (x_1 + (x_1^2 + x_2^2)^{1/2}x_2,\ x_2 - (x_1^2 + x_2^2)^{1/2}x_1)$. Then we have $\omega(f) = 1$ and $|f| = \infty$.

Notice that even though f is a function defined on X_0, the normal component f_ν of f

$$f_\nu(x, x^*) = \frac{\langle f(x), x^*\rangle}{\|x\|\ \|x^*\|}\, x,$$

is actually defined on Π_0. This is one of the reasons why we studied in Section 2 the more general maps $F\colon \Pi_0 \to X$.

Of course, this ambiguity disappears if X is a Banach space with a smooth unit ball. Since, in this case, there is a unique semi-inner product $[\cdot, \cdot]$ in X such that $[x, x] = \|x\|^2$, $x \in X$; and the formulas for $\omega(f)$, $\alpha(f)$, $\Omega(f)$, f_ν and f_τ, for a given $f \in W(X)$, take the form

$$\omega(f) = \limsup_{\|x\|\to\infty} \frac{|[f(x), x]|}{\|x\|^2}, \qquad \alpha(f) = \liminf_{\|x\|\to\infty} \frac{|[f(x), x]|}{\|x\|^2}$$

$$\Omega(f) = \bigcap_{r>0} \overline{\phi_f(E_r)}, \qquad \text{where} \quad E_r = \{x \in X;\ \|x\| \geqslant r\} \qquad (r > 0),$$

$$\phi_f(x) = \frac{[f(x), x]}{\|x\|^2} \qquad (x \neq 0),$$

$$f_\nu(x) = \phi_f(x)x \qquad \text{and} \qquad f_\tau(x) = f(x) - f_\nu(x) \qquad (x \neq 0).$$

Now, the following should be obvious.

PROPOSITION 4.1. *If* $T \in L(X)$. *Then*:

(a) $\Omega(T) = \overline{V(T)}$.

(b) $\omega(T) = v(T)$.

To end this section we shall give the following diagram, which relates the different vector spaces that we have dealt with. The arrows represent the obvious linear maps defined between them, and this is a commutative diagram.

$$\begin{tikzcd}
L(X) \arrow[d] \arrow[dr] & & \\
B(X) \arrow[r] \arrow[d] \arrow[dr] & B_*(X) \arrow[d] \arrow[dr] & \\
Q(X) \arrow[r] \arrow[d] \arrow[dr] & Q_*(X) \arrow[r] \arrow[d] \arrow[dr] & \tilde{Q}_*(X) \arrow[d] \\
W(X) \arrow[r] & W_*(X) \arrow[r] & \hat{W}_*(X)
\end{tikzcd}$$

5. THE *-ASYMPTOTIC SPECTRUM

In [5] Furi and Vignoli defined the "asymptotic spectrum" $\Sigma(f)$ of a quasi-bounded map f on X, as the set

$$\Sigma(f) = \{\lambda \in \mathbb{K};\, d(\lambda I - f) = 0\},$$

where

$$d(f) = \liminf_{\|x\| \to \infty} \frac{\|f(x)\|}{\|x\|}.$$

Among other things, they showed that $\Sigma(f)$ is compact; that if $T \in L(X)$, then $\Sigma(T)$ is the approximate point spectrum of T, and if T is also compact, then $\Sigma(T)$ is precisely the spectrum of T. In this section we are going to define the "*-asymptotic spectrum" $\Sigma_*(F)$ for $F \in Q_*(X)$, and study its relations with the *-numerical range $\Omega_*(F)$.

DEFINITION 5.1. For any $F \in Q_*(X)$ we define

$$d_*(F) = \lim_{r \to \infty} \inf_{\Pi_r} \frac{\|F(x, x^*)\|}{\|x\|},$$

and the $*$-asymptotic spectrum $\Sigma_*(F)$ of F, as the set

$$\Sigma_*(F) = \{\lambda \in \mathbb{K};\ d_*(\lambda\pi - F) = 0\}$$

Where, as usual, π denotes the natural projection of $X \times X^*$ onto X.

PROPOSITION 5.1. *If $F, G \in Q_*(X)$ and $\mu \in \mathbb{K}$, then:*

(a) $0 \leqslant d_*(F) \leqslant |F|_*$.

(b) $d_*(\mu F) = |\mu| \, d_*(F)$.

(c) $d_*(F + G) \leqslant d_*(F) + |G|_*$.

(d) $d_*(F) - |G|_* \leqslant d_*(F + G)$.

(e) $|d_*(F) - d_*(G)| \leqslant |F - G|_*$. *So d_* is actually defined in $\tilde{Q}_*(X)$.*

(f) $d_*(F) \leqslant |\lambda|$, $\lambda \in \Sigma_*(F)$.

Proof. It is analogous to the one given in Proposition 3.7, so it will be omitted. See also Proposition 2.1 in [5].

PROPOSITION 5.2. *If $F, G \in Q_*(X)$ and $\mu \in \mathbb{K}$, then:*

(a) $\Sigma_*(F) \subseteq \Omega_*(F)$.

(b) *If* $|F - G|_* = 0$, *then* $\Sigma_*(F) = \Sigma_*(G)$.

(c) $r_*(F) \leqslant |F|_*$, *where*

$$r_*(F) = \sup\{|\lambda|;\ \lambda \in \Sigma_*(F)\},$$

is the "$*$-asymptotic spectral radius" of F.

(d) $\Sigma_*(F)$ *is compact.*

(e) $\Sigma_*(\mu F) = \mu \Sigma_*(F)$.

(f) $\Sigma_*(\mu\pi + F) = \mu + \Sigma_*(F)$.

Proof. (a) It follows from the obvious inequality $\alpha_*(F) \leqslant d_*(F)$ and Proposition 3.5.

(b) Immediate from Proposition 5.1(e).

(c) Let $\lambda \in \Sigma_*(F)$. By Proposition 5.1(d) we have $|\lambda| - |F|_* \leqslant d_*(\lambda\pi - F) = 0$.

(d) By Proposition 5.1(e), the mapping $\lambda \to d_*(\lambda\pi - F)$ is continuous, and hence $\Sigma_*(F)$ is closed. By (c) it is bounded and hence compact.

Recall that a Banach space X is said to be uniformly convex, if whenever $x_n \in X$, $y_n \in X$, $\|x_n\| \leqslant 1$, $\|y_n\| \leqslant 1$ and $\|x_n + y_n\| \to 2$, then $\|x_n - y_n\| \to 0$.

PROPOSITION 5.3. *If X is uniformly convex and $F \in Q_*(X)$, then*

$$\{\lambda \in \Omega_*(F);\ |\lambda| = |F|_*\} \subseteq \Sigma_*(F). \tag{1}$$

Proof. Let $\lambda \in \Omega_*(F)$ and $|\lambda|_* = |F|_*$. We may assume that $\lambda \neq 0$, for otherwise $\tilde{F} = 0$ and the result follows immediately. Since we may replace F by $\lambda^{-1}F$, there is no loss of generality in assuming that $|F|_* = \lambda = 1$.

Now, there exists $(x_n, x_n^*) \in \Pi_n$ such that

$$\frac{\langle F(x_n, x_n^*), x_n^* \rangle}{\|x_n\|\,\|x_n^*\|} \to 1,$$

and therefore

$$\frac{\langle (\pi + F)(x_n, x_n^*), x_n^* \rangle}{\|x_n\|\,\|x_n^*\|} \to 2. \tag{2}$$

Since

$$\begin{aligned} 1 + \frac{\|F(x_n, x_n^*)\|}{\|x_n\|} &\geqslant \frac{\|(\pi + F)(x_n, x_n^*)\|}{\|x_n\|} \\ &\geqslant \frac{|\langle (\pi + F)(x_n, x_n^*), x_n^* \rangle|}{\|x_n\|\,\|x_n^*\|}, \end{aligned} \tag{3}$$

and $|F|_* = 1$, it follows that

$$\|x_n/\|x_n\| + F(x_n, x_n^*)/\|x_n\|\| \to 2. \tag{4}$$

But (4) and X uniformly convex imply

$$\frac{\|(\pi - F)(x_n, x_n^*)\|}{\|x_n\|} \to 0. \tag{5}$$

Hence from (5) we obtain $d_*(\pi - F) = 0$, i.e., $1 \in \Sigma_*(F)$.

6. A NONLINEAR VERSION OF LUMER'S FORMULA

Our aim in this section is to prove a nonlinear version of Lumer's formula (4) in Section 1 for the class of Lipschitz maps (Proposition 6.3). But before we do this, we are going to state an elementary result which is a generalization of the well known properties of the logarithmic norm for bounded linear operators on a Banach space X.

PROPOSITION 6.1. *Let X be a Banach space, and let Φ be a vector space of continuous maps $f\colon X_0 \to X$ such that $I \in \Phi$. Let σ be a semi-norm defined on Φ such that $\sigma(I) = 1$. If for every $f \in \Phi$ we define*

$$\sigma'(f) = \lim_{\rho \to 0+} \frac{\sigma(I + \rho f) - 1}{\rho}, \tag{1}$$

then the limit (1) *exists and satisfies the properties*:

(a) $|\sigma'(f)| \leqslant \sigma(f)$.
(b) $\sigma'(\mu f) = \mu \sigma'(f)$, $\mu \geqslant 0$.
(c) $\sigma'(f + g) \leqslant \sigma'(f) + \sigma'(g)$.
(d) $|\sigma'(f) - \sigma'(g)| \leqslant \sigma(f - g)$.

Proof. That the limit (1) exists, follows from the fact that $\sigma(I + \rho f)$, $0 \leqslant \rho < \infty$, is a convex function of ρ. Properties (a)–(d) are immediate from the fact that σ is a seminorm on Φ.

PROPOSITION 6.2. *If $f \in W(X)$, then*

$$\sup \operatorname{Re} \Omega(f) \leqslant \omega'(f). \tag{1}$$

Proof. From the inequality

$$\operatorname{Re} \frac{\langle f(x), x^* \rangle}{\|x\| \, \|x^*\|} \leqslant \frac{1}{\rho} \left\{ \frac{|\langle x + \rho f(x), x^* \rangle|}{\|x\| \, \|x^*\|} - 1 \right\}, \qquad \rho > 0;$$

and the obvious fact

$$\sup \operatorname{Re} \Omega(f) = \lim_{r \to \infty} \sup_{\Pi_r} \operatorname{Re} \frac{\langle f(x), x^* \rangle}{\|x\| \, \|x^*\|},$$

we obtain

$$\sup \operatorname{Re} \Omega(f) \leqslant \frac{\omega(I + \rho f) - 1}{\rho}, \qquad \rho > 0. \tag{2}$$

Now, (1) follows, if in (2) we let $\rho \to 0+$.

PROPOSITION 6.3. *If $f\colon X \to X$ is a Lipschitz map, i.e., there exists $k > 0$ such that*

$$\|f(x) - f(y)\| \leqslant k\|x - y\|, \; x, y \in X. \tag{1}$$

Then

$$\sup \operatorname{Re} \Omega(f) = \omega'(f) = |f|'.$$

Proof. Since, clearly $\omega'(f) \leqslant |f|'$, from the previous proposition we see that it suffices to show that

$$|f|' \leqslant \sup \operatorname{Re} \Omega(f). \tag{3}$$

Let $\mu = \sup \operatorname{Re} \Omega(f)$ and $\mu_r = \sup \operatorname{Re} \overline{\phi_f(\Pi_r)}$ $(r > 0)$, where ϕ_f is as in Definition 3.1.

We have for $(x, x^*) \in \Pi_r$ $(r > 0)$

$$\begin{aligned}\frac{\|(I - \rho f)(x)\|}{\|x\|} &\geqslant \left| \frac{\langle (I - \rho f)(x), x^* \rangle}{\|x\| \, \|x^*\|} \right| \\ &\geqslant \left| 1 - \rho \frac{\langle f(x), x^* \rangle}{\|x\| \, \|x^*\|} \right| \\ &\geqslant 1 - \rho \operatorname{Re} \frac{\langle f(x), x^* \rangle}{\|x\| \, \|x^*\|} \\ &\geqslant 1 - \rho \sup \operatorname{Re} \overline{\phi_f(\Pi_r)} \\ &\geqslant 1 - \rho\mu_r ,\end{aligned}$$

and using the fact $\lim_{r\to\infty} \mu_r = \mu$, we obtain

$$\frac{\|(I - \rho f)(x)\|}{\|x\|} \geqslant 1 - \rho\mu_r > 0, \qquad \|x\| \geqslant r, \tag{4}$$

for all $\rho > 0$ sufficiently small.

If we apply (1) we obtain

$$\begin{aligned}\|x + \rho f(x)\| &\geqslant \|x\| - \rho\|f(x)\| \\ &\geqslant \|x\| - \rho(\|f(0)\| + k\|x\|) \\ &\geqslant (1 - k\rho)\|x\| - \rho\|f(0)\|.\end{aligned}$$

Thus, if we let $0 < \rho < 1/k$, we see from this last inequality that we can choose $\|x\| \geqslant r$ large enough so that

$$\|x + \rho f(x)\| \geqslant r.$$

Hence we can apply (4) with $x + \rho f(x)$ instead of x and obtain

$$\|(I - \rho f)(I + \rho f)(x)\| \geqslant (1 - \rho\mu_r)\|x + \rho f(x)\|,$$

and

$$\|(I + \rho f)(x) - \rho f(I + \rho f)(x)\| \geqslant (1 - \rho\mu_r)\|x + \rho f(x)\|. \tag{5}$$

From (1) we obtain

$$\begin{aligned}\|(I+\rho f)(x)-\rho f(I+\rho f)(x)\| &\leqslant \|x\|+\rho\|f(x)-f((I+\rho f)(x))\| \\ &\leqslant \|x\|+\rho k\|x-(I+\rho f)(x)\| \\ &\leqslant \|x\|+\rho^2 k\|f(x)\|.\end{aligned}$$

Thus we have

$$\|(I+\rho f)(x)-\rho f(I+\rho f)(x)\| \leqslant \|x\|+\rho^2 k\|f(x)\|. \tag{6}$$

From (5) and (6) we get

$$\|x\|+\rho^2 k\|f(x)\| \geqslant (1-\rho\mu_r)\|x+\rho f(x)\|,$$

and hence

$$1+\rho^2 k\frac{\|f(x)\|}{\|x\|} \geqslant (1-\rho\mu_r)\frac{\|x+\rho f(x)\|}{\|x\|} \tag{7}$$

If in (7) we take the lim sup as $r\to\infty$ we obtain

$$1+\rho^2 k\,|f| \geqslant (1-\rho\mu)|I+\rho f|,$$

and

$$\frac{|I+\rho f|-1}{\rho} \leqslant \frac{\rho k\,|f|+\mu}{1-\rho\mu}. \tag{8}$$

If in (8) we let $\rho\to 0+$, we obtain (3), and this completes the proof.

7. The Numerical Range of the Adjoint

If Y is a Banach space, then Y_s^* and Y_w^* will denote the dual of Y together with the norm (strong) and weak* topologies respectively. We denote by

$$J\colon X\to (X_s^*)_w^*$$

the canonical isometric embedding of X into its bidual $(X_s^*)_w^*$. A well known result of Goldstine [10] asserts that $J(B_R)$ is weak*-dense in B_R^{**}, where $B_R=\{x\in X;\ \|x\|\leqslant R\}$ and $B_R^{**}=\{x^{**}\in(X_s^*)_w^*;\ \|x^{**}\|\leqslant R\}$ $(R>0)$. Since our objective in this section is to study maps from $X_s^*\times(X_s^*)_w^*$ into X_s^*, we define the following sets

$$\begin{aligned}\Pi_r^* = \{(x^*, x^{**})\in X_s^*\times(X_s^*)_w^*;\ &\|x^*\|=\|x^{**}\|\geqslant r, \\ &\|x^*\|^2=\langle x^*, x^{**}\rangle\} \qquad (r>0),\end{aligned}$$

and

$$\Pi_0^* = \bigcup_{r>0} \Pi_r^*.$$

As before, we shall assume that Π_0^* has the norm $\times$ weak* topology induced as a subset of $X_s^* \times (X_s^*)_w^*$. From Proposition 2.1 we know that each Π_w^* $(r > 0)$ and Π_0^* are connected subsets of $X_s^* \times (X_s^*)_w^*$. Note that each Π_r can be considered, in a natural way, as a subset of Π_r^* by means of the identification $x = J(x)$ and

$$\Pi_r \leftrightarrow \Pi_r^{-1} = \{(x^*, x); (x, x^*) \in \Pi_r\} \subseteq \Pi_r^*.$$

Thus, $G \in W_*(X^*)$ will mean that $G: \Pi_0^* \to X^*$ is a continuous map such that

$$\omega_*(G) = \lim_{r\to\infty} \sup_{\Pi_r^*} \frac{|\langle G(x^*, x^{**}), x^{**}\rangle|}{\|x^*\| \, \|x^{**}\|} < \infty.$$

DEFINITION 7.1. If $G \in W_*(X^*)$, we define the "lower *-numerical range" $\Lambda\Omega_*(G)$ of G as the set

$$\Lambda\Omega_*(G) = \bigcap_{r>0} \overline{\psi_G(\Pi_r^{-1})}.$$

where

$$\psi_G(x^*, x^{**}) = \frac{\langle G(x^*, x^{**}), x^{**}\rangle}{\|x^*\| \, \|x^{**}\|}, \qquad (x^*, x^{**}) \in \Pi_0^*,$$

and

$$\psi_G(x^*, Jx) = \frac{\langle G(x^*, Jx), Jx\rangle}{\|x^*\| \, \|Jx\|}$$

$$= \frac{\langle x, G(x^*, Jx)\rangle}{\|x\| \, \|x^*\|}, \qquad (x, x^*) \in \Pi_0.$$

It is clear that

$$\Lambda\Omega_*(G) \subseteq \Omega_*(G),$$

where, as usual, $\Omega_*(G)$ denotes the *-numerical range of G.

We are going to see that actually one has

$$\Lambda\Omega_*(G) = \Omega_*(G).$$

But first we need the following result.

LEMMA 7.1. *Π_r^{-1} is norm $\times$ weak* dense in Π_r^* $(r > 0)$, i.e., in the topology of $X_s^* \times (X_s^*)_w^*$.*

Proof. We have to show that given $(x_0^*, x_0^{**}) \in \Pi_r^*$, there exists a sequence $\{(y_n, y_n^*)\}$ in Π_r such that

$$\|y_n^* - x_0^*\| \to 0 \qquad \text{and} \qquad y_n \rightharpoonup x_0^{**} \quad \text{(weakly)} \tag{1}$$

Now, from Goldstine's theorem we know that there is a sequence $\{x_n\}$ in X such that

$$\|x_n\| \leqslant \|x_0^{**}\| = \|x_0^*\| \qquad \text{and} \qquad x_n \rightharpoonup x_0^{**}. \tag{2}$$

In particular we have

$$|\langle x_n, x_0^* \rangle - \langle x_0^*, x_0^{**} \rangle| \to 0. \tag{3}$$

Thus we can find a subsequence of $\{x_n\}$, which will be denoted in the same way $\{x_n\}$, such that

$$|\langle x_n, x_0^* \rangle - \langle x_0^*, x_0^{**} \rangle| < (1/2n)^2 \|x_0^*\|^2,$$

and hence from (2) we have (recall $r^2 \leqslant \|x_0^*\|^2 = \langle x_0^*, x_0^{**} \rangle$)

$$|1 - \langle x_n/\|x_0^*\|, x_0^*/\|x_0^*\| \rangle| < (1/2n)^2 \qquad \text{and} \qquad \|x_n\|/\|x_0^*\| \leqslant 1. \tag{4}$$

From (4) and the Bishop–Phelps–Bollobás Theorem [3], we see that for each n there exists $u_n \in X$, $u_n^* \in X^*$ such that $\|u_n\| = \|u_n^*\| = \langle u_n, u_n^* \rangle = 1$ and

$$\|u_n - x_n/\|x_0^*\|\| < 1/n, \qquad \|u_n^* - x_0^*/\|x_0^*\|\| < 1/n. \tag{5}$$

If we let $y_n = \|x_0^*\| u_n$ and $y_n^* = \|x_0^*\| u_n^*$, then

$$\|y_n\| = \|y_n^*\| = \|x_0^*\| \geqslant r, \qquad \|x_0^*\|^2 = \langle y_n, y_n^* \rangle,$$

i.e., $\{(y_n, y_n^*)\} \subseteq \Pi_r$.

Now, from (5) we get

$$\|y_n - x_n\| < n^{-1}\|x_0^*\| \qquad \text{and} \qquad \|y_n^* - x_0^*\| < n^{-1}\|x_0^*\|. \tag{6}$$

Hence

$$\|y_n^* - x_0^*\| \to 0.$$

Rest to show that

$$y_n \rightharpoonup x_0^{**}.$$

We have for each $x^* \in X^*$

$$\begin{aligned} |\langle y_n, x^* \rangle - \langle x^*, x_0^{**} \rangle| &\leqslant |\langle y_n, x^* \rangle - \langle x_n, x^* \rangle| + |\langle x_n, x^* \rangle - \langle x^*, x_0^{**} \rangle| \\ &\leqslant \|y_n - x_n\| \|x^*\| + |\langle x_n, x^* \rangle - \langle x^*, x_0^{**} \rangle| \end{aligned}$$

From (2) and (6) we see that

$$|\langle y_n, x^*\rangle - \langle x^*, x_0^{**}\rangle| \to 0,$$

and this completes the proof.

PROPOSITION 7.1. *If $G \in W_*(X^*)$, then $\Lambda\Omega_*(G) = \Omega_*(G)$.*

Proof. We have only to prove that $\Omega_*(G) \subseteq \Lambda\Omega_*(G)$. Since $\psi_G: \Pi_0^* \to \mathbb{K}$ is a continuous map we have

$$\psi_G(\overline{\Pi_r^{-1}}) \subseteq \overline{\psi_G(\Pi_r^{-1})} \qquad (r > 0). \tag{1}$$

From the previous lemma

$$\psi_G(\Pi_r^*) \subseteq \psi_G(\overline{\Pi_r^{-1}}) \qquad (r > 0). \tag{2}$$

Combining (1) and (2) we obtain

$$\Omega_*(G) = \bigcap_{r>0} \overline{\psi_G(\Pi_r^*)} \subseteq \bigcap_{r>0} \overline{\psi_G(\Pi_r^{-1})} = \Lambda\Omega_*(G).$$

DEFINITION 7.2. Let $f \in W(X)$. If there exists $g \in W(X^*)$ such that

$$\langle f(x), x^*\rangle = \langle x, g(x^*)\rangle, \ (x, x^*) \in \Pi_0 . \tag{1}$$

Then we say that g is an adjoint of f. Note that, in this case, we have

$$\Lambda\Omega(g) = \Omega(f). \tag{2}$$

One could be tempted to define an adjoint of f, by imposing the condition

$$\langle f(x), x^*\rangle = \langle x, g(x^*)\rangle, \ x \in X, \ x^* \in X^*.$$

But, as one can easily see, this condition would imply that both f and g must be linear. Of course, if $T \in L(X)$, then its (unique) Banach-adjoint $T^* \in L(X^*)$ satisfies condition (1) above.

That an adjoint, in the nonlinear case, does not have to be unique, it is shown in the following:

PROPOSITION 7.2. *Let $f \in W(X)$. If $g, h \in W(X^*)$ are adjoints of f, then $\omega(g - h) = 0$.*

Proof. We have

$$\langle x, g(x^*)\rangle = \langle f(x), x^*\rangle = \langle x, h(x^*)\rangle, \ (x, x^*) \in \Pi_0 .$$

Hence

$$\langle x, (g - h)(x^*)\rangle = 0, (x, x^*) \in \Pi_0 .$$

This last fact together with Proposition 7.1 show that

$$\Omega(g - h) = \Lambda\Omega(g - h) = \{0\},$$

and hence that $\omega(g - h) = 0$.

Next we state the main result of this section

PROPOSITION 7.3. *Let $f \in W(X)$. If $g \in W(X^*)$ is an adjoint of f, then $\Omega(g) = \Omega(f)$.*

Proof. Immediate from Proposition 7.1 and the identity $\Lambda\Omega(g) = \Omega(f)$.

8. CALCULUS OF DEGREE FOR NUMERICALLY BOUNDED MAPS

In [5] Furi and Vignoli obtained some surjectivity results for compact quasi-bounded maps. Their main result is the following

PROPOSITION. (Furi–Vignoli). *Let $f \in Q(X)$ be compact and $\lambda \neq 0$. If λ belongs to the unbounded component of $\mathbb{K} - \Sigma(f)$, then $\lambda I - f$ is onto*

Now, from Proposition 5.2(a) we know that $\Sigma(f) \subseteq \Omega(f)$. That, in general, this inclusion is proper can be seen by looking at the linear case. The main result of this section is the following

PROPOSITION 8.1. *Let $f \in W(X)$ be compact and $\lambda \neq 0$. If λ belongs to the unbounded component of $\mathbb{K} - \Omega(f)$, then $\lambda I - f$ is onto.*

It is clear from the inclusions

$$\Sigma(f) \subset \Omega(f) \qquad \text{and} \qquad Q(X) \subset W(X),$$

that our result and the one by Furi–Vignoli are complementary. In the sense that, for obtaining surjectivity results, they have more freedom of choice with the scalars $\lambda \neq 0$, and we with the functions f.

The technique used here for obtaining surjectivity results for numerically bounded compact maps, is essentially the one developed in [5], except for minor modifications.

Let $f \in W(X)$ be compact and such that $\alpha(I - f) > 0$. From the definition of α it follows that there exists $r_0 > 0$ such that $x \neq f(x)$ for $\|x\| \geqslant r_0$. This implies that for $r > r_0$ the Leray–Schauder degree $\deg(I - f, B_r, 0)$ for the compact perturbation of the identity $I - f$, restricted to the closed ball $\bar{B}_r$ of radius $r > 0$ and centered at the origin, is defined. Since $\deg(I - f, B_r, 0) = \deg(I - f, B_s, 0)$ for any $r, s > r_0$, we define

$$\deg(I - f) = \lim_{r \to \infty} \deg(I - f, B_r, 0).$$

Let f be as above and $\lambda \in \mathbb{K}$. As in [5] we say that $\lambda I - f$ is admissible for surjectivity (s-admissible) if $\lambda \neq 0$ and $\alpha(\lambda I - f) > 0$. We set

$$\deg(\lambda I - f) = \deg(I - \lambda^{-1} f).$$

A homogopy Φ: $X \times [0, 1] \to X$ is said to be an s-homotopy if the following conditions are satisfied:

(a) $\Phi(x, t) = \lambda(t)x - \phi(x, t)$, where λ:$[0, 1] \to \mathbb{K}$ and ϕ:$X \times [0, 1] \to X$ are continuous.

(b) ϕ is uniformly numerically bounded at ∞ with respect to t, in the sense

$$\frac{|\langle \phi(x, t) - \phi(x, t_0), x^* \rangle|}{\|x\| \; \|x^*\|} \to 0,$$

when $\|x\| \to \infty$ and $t \to t_0$, where $(x, x^*) \in \Pi_0$.

(c) For any bounded set $A \subset X$, the set $\phi(A \times [0, 1])$ is relatively compact.

(d) The mapping $\lambda(t)I - \phi(\cdot, t)$ is s-admissible for any $t \in [0, 1]$.

Properties (a)–(c) are obviously satisfied when $\phi(x, t)$ is of the form

$$\phi(x, t) = \sum_{j=1}^{n} \lambda_j(t) f_j(x),$$

where λ_j: $[1, 1] \to \mathbb{K}$ is continuous and $f_j \in W(X)$ is compact for $j = 1, 2, \ldots, n$.

Two s-admissible maps are said to be s-homotopic if there exists an s-homotopy joining them.

The following result is the analoge of Proposition 3.1 in [5].

PROPOSITION 8.2. $\deg(\lambda I - f)$ *has the following properties*:

(a) *Two s-homotopic mappings have the same degree.*

(b) *Let $f, g \in W(X)$ be compact. If $\alpha(\lambda I - f) > \omega(f - g)$, then $\lambda I - f$ and $\lambda I - g$ are homotopic, and hence* $\deg(\lambda I - f) = \deg(\lambda I - g)$.

(c) *If $\lambda I - f$ is s-admissible and $\omega(f - g) = 0$, then $\lambda I - g$ is s-admissible and* $\deg(\lambda I - f) = \deg(\lambda I - g)$.

(d) *Let $f \in W(X)$ be compact. If λ_1, λ_2 are different from zero and belong to the same component of $K - \Omega(f)$, then* $\deg(\lambda_1 I - f) = \deg(\lambda_2 I - f)$.

(e) *If* $\deg(\lambda I - f) \neq 0$, *then $\lambda I - f$ is onto*

(f) $\deg(\lambda I - 0) = 1$, $\lambda \neq 0$.

Proof. (a) Let $\Phi(x, t) = \lambda(t)x - \phi(x, t)$ be an s-homotopy joining $\lambda_0 I - f_0$ and $\lambda_1 I - f_1$. We have to show that there exists $r > 0$ such that the equation $\lambda(t)x - \phi(x, t) = 0$ has no solutions for $\|x\| \geqslant r$ and $t \in [0, 1]$. Because, if this is the case, the map $\Phi_r \colon \bar{B}_r \times [0, 1] \to X$ defined by $\Phi_r(x, t) = x - \lambda(t)^{-1}\phi(x, t)$ is an admissible homotopy for the Leray–Schauder degree, and hence

$$\deg(\lambda(t)I - \phi(\cdot, t)) = \deg(I - \lambda(t)^{-1}\phi(\cdot, t), B_r, 0) = \text{const.} \quad \text{for all} \quad t \in [0, 1].$$

Suppose that for every integer $n \geqslant 1$ the equation $\lambda(t)x - \phi(x, t) = 0$, has a solution (x_n, t_n) such that $\|x_n\| \geqslant n$. Pick for each n an $x_n^* \in X^*$ such that $(x_n, x_n^*) \in \Pi_n$. Since $\{t_n\} \subset [0, 1]$ we may assume that $t_n \to t_0 \in [0, 1]$. We have

$$\begin{aligned}
\frac{\langle \lambda(t_0)\, x_n - \phi(x_n, t_0), x_n^* \rangle}{\|x_n\|\, \|x_n^*\|}
&= \frac{\langle \lambda(t_0)\, x_n - \phi(x_n, t_0) - \lambda(t_n)\, x_n + \phi(x_n, t_n), x_n^* \rangle}{\|x_n\|\, \|x_n^*\|} \\
&= (\lambda(t_0) - \lambda(t_n)) + \frac{\langle \phi(x_n, t_n) - \phi(x_n, t_0), x_n^* \rangle}{\|x_n\|\, \|x_n^*\|}
\end{aligned}$$

$$\begin{aligned}
\frac{|\langle \lambda(t_0)\, x_n - \phi(x_n, t_0), x_n^* \rangle|}{\|x_n\|\, \|x_n^*\|}
&\leqslant |\lambda(t_0) - \lambda(t_n)| + \frac{|\langle \phi(x_n, t_n) - \phi(x_n, t_0), x_n^* \rangle|}{\|x_n\|\, \|x_n^*\|},
\end{aligned}$$

and so

$$\alpha(\lambda(t_0)I - \phi(\cdot, t_0)) \leqslant \lim_{\substack{\|x\| \to \infty \\ t \to t_0}} \frac{|\langle \phi(x, t) - \phi(x, t_0), x^* \rangle|}{\|x\|\, \|x^*\|} = 0.$$

Therefore $\alpha(\lambda(t_0)I - \phi(\cdot, t_0)) = 0$, which contradicts the s-admissibility of $\lambda(t_0)I - \phi(\cdot, t_0)$.

(b) Define a homotopy joining $\lambda I - f$ and $\lambda I - g$ as

$$\Phi(x, t) = \lambda x - ((1 - t)f(x) + tg(x)).$$

Clearly this is continuous and satisfies conditions (b) and (c) of s-homotopy. Therefore we have only to show that it also satisfies property (d). By Proposition 3.7 we get

$$\alpha(\lambda I - (1 - t)f - tg) = \alpha(\lambda I - f + t(f - g)) \geqslant \alpha(\lambda I - f) - t\omega(f - g) > 0, \quad \text{for} \quad t \in [0, 1].$$

(c) Follows immediately from (b) since $\alpha(\lambda I - f) > 0$ and $\omega(f - g) = 0$.

(d) Let Λ be the component of $\mathbb{K} - \Omega(f)$ containing λ_1 and λ_2. Since Λ is open there exists a path $\lambda:[0, 1] \to \Lambda - \{0\}$ joining λ_1 and λ_2. Clearly $\lambda_1 I - f$ and $\lambda_2 I - f$ are s-homotopic via the map $\Phi(x, t) = \lambda(t)x - f(x)$. The assertion follows from (a).

(e) We want to prove that the map f_p: $X \to X$ defined by $f_p(x) = \lambda^{-1}f(x) + p)$, has a fixed point for any $p \in X$. By property (c), $\deg(\lambda I - f) = \deg(I - f_p)$, since $\omega(\lambda^{-1}f - f_p) = 0$. On the other hand there exists $r > 0$ such that $\deg(I - f_p, B_r, 0) = \deg(I - f_p)$. Therefore $\deg(I - f_p, B_r, 0) = \deg(\lambda I - f) \neq 0$, and we are done.

(f) Obvious.

Now we prove a more general form of Proposition 8.1

PROPOSITION 8.3. *Let $f, g \in W(X)$ be compact. Assume*

$$\alpha(\lambda I - f) > \omega(f - g),$$

where $\lambda \neq 0$ belongs to the unbounded component of $K - \Omega(f)$. Then $\deg(\lambda I - f) = 1$, *and hence $\lambda I - f$ is onto.*

Proof. By Proposition 8.2(b) it is enough to prove that $\deg(\lambda I - g) = 1$. On the basis of Proposition 8.2(d) it suffices to show that $\deg(rI - g) = 1$, when r is any real number greater than $\omega(f)$. Define the homotopy $\Phi(x, t) = rx - tg(x)$, $0 \leqslant t \leqslant 1$. Clearly Φ is a homotopy joining $rI - g$ with rI. Moreover

$$0 < r - t\omega(f) \leqslant \alpha(rI - tg), 0 \leqslant t \leqslant 1.$$

Therefore Φ is an s-homotopy. By Proposition 8.2(a)–(f) we get $\deg(rI - g) = 1$.

9. THE NUMERICAL RANGE FOR VECTOR FIELDS ON THE UNIT SPACE

Let X be a Banach space and $S = \{x \in X; \|x\| = 1\}$ be the unit sphere in X. Let Φ: $S \to X$ be a continuous map on S, i.e., a "vector field" on S. We say that Φ is numerically bounded, if the map

$$\tilde{\Phi}(x) = \|x\|\Phi(\|x\|^{-1}x), \qquad x \neq 0,$$

is numerically bounded. In this case we let $\omega(\Phi) = \omega(\tilde{\Phi})$, $\alpha(\Phi) = \alpha(\tilde{\Phi})$ and $\Omega(\Phi) = \Omega(\tilde{\Phi})$.

If we set

$$\Pi = \{(u, u^*) \in X \times X^*; \|u\| = \|u^*\| = \langle u, u^*\rangle = 1\},$$

then an analogous proof as the one given in Proposition 2.1 shows that Π is a connected subset of $X \times X^*$ with the norm $\times$ weak* topology (see also Theorem 11.4 in [2]).

PROPOSITION 9.1. *Let Φ be a numerically bounded vector field on S. Then*:

(a) $\omega(\Phi) = \sup_{\Pi} |\langle \Phi(u), u^* \rangle|$.

(b) $\alpha(\Phi) = \inf_{\Pi} |\langle \Phi(u), u^* \rangle|$.

(c) $\Omega(\Phi) = \{\langle \Phi(u), u^* \rangle; (u, u^*) \in \Pi\}^-$.

Proof. (a) and (b) follow from

$$\frac{\langle \tilde{\Phi}(x), x^* \rangle}{\|x\| \, \|x^*\|} = \frac{\langle \|x\| \, \Phi(\|x\|^{-1}x), x^* \rangle}{\|x\| \, \|x^*\|} = \langle \Phi(\|x\|^{-1}x), \|x^*\|^{-1}x^* \rangle$$

$$= \langle \Phi(u), u^* \rangle,$$

where $u = \|x\|^{-1}x$, $u^* = \|x^*\|^{-1}x^*$ and $(u, u^*) \in \Pi$. Now (c) becomes evident.

From this last result we see that $\Omega(\Phi)$ coincides with the closure $\overline{V(\Phi)}$ of the numerical range $V(\Phi)$ of a continuous map Φ: $S \to X$ as defined by Bonsall, Cain and Schneider [2].

As an application of the previous consider the following problem: From the identity $x = \|x\|(\|x\|^{-1}x)$, $x \neq 0$, we see that

$$\mathbb{R}^+ I(S) = X,$$

where $\mathbb{R}^+ = [0, \infty[$ and I: $X \to X$ is the identity map. The question now is; under which continuous maps Ψ: $S \to X$ we still have

$$\mathbb{R}^+(I + \Psi)(S) = X?$$

A partial answer is given by the following:

PROPOSITION 9.2. *If Φ is a numerically bounded compact vector field on S, then*

$$\mathbb{R}^+(I - \mu\Phi)(S) = X \qquad \text{for} \quad |\mu| < \frac{1}{\omega(\Phi)}. \tag{1}$$

Proof. Since Φ is compact, then it is bounded in S; and hence if we define $\tilde{\Phi}(x) = \|x\|\Phi(\|x\|^{-1}x)$, $x = 0$ and $\tilde{\Phi}(0) = 0$. Then $\tilde{\Phi}$ is a compact numerically bounded map on X. From Proposition 8.1 we see that if λ belongs to the unbounded component of $\mathbb{K} - \Omega(\Phi)$, then the map $\lambda I - \tilde{\Phi}$ is onto. Thus, if $|\lambda| > \omega(\Phi)$, then $\lambda I - \tilde{\Phi}$ is onto. Hence if we let $\mu = \lambda^{-1}$, then $I - \mu\tilde{\Phi}$ is onto for $|\mu| < \omega(\Phi)^{-1}$.

Let $|\mu| < \omega(\Phi)^{-1}$. If $y \in X$, $y \neq 0$, then there is an $x \in X$, $x \neq 0$ such that $x - \mu\tilde{\Phi}(x) = y$, and hence $x/\|x\| - \mu\Phi(x/\|x\|) = y/\|x\|$. Thus if we let

$u = \|x\|^{-1}x \in S$ and $\beta = \|x\|$ we see that $\beta(u - \mu\Phi(u)) = y$, and this completes the proof.

COROLLARY 9.1. *Let H be a Hilbert space and Φ be a compact tangent vector field on S, i.e.,*

$$\|\Phi(u)\| \leqslant 1,\ (\Phi(u) \mid u) = 0, \qquad u \in S.$$

Then

$$\mathbb{R}^+(I - \mu\Phi)(S) = H \qquad \text{for} \quad \mu \in \mathbb{K}.$$

Proof. Immediate from the previous proposition, because our hypothesis implies that $\omega(\Phi) = 0$.

ACKNOWLEDGMENT

We would like to thank A. Vignoli for stimulating conversations on some of the topics of this paper.

REFERENCES

1. F. L. BAUER, On the field of values subordinate to a norm, *Numer. Math.* **4** (1962), 103–111.
2. F. F. BONSALL AND J. DUNCAN, "Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras," Cambridge University Press, London/New York, 1971.
3. F. F. BONSALL AND J. DUNCAN, "Numerical Ranges II," Cambridge University Press, London/New York, 1973.
4. M. FURI AND A. VIGNOLI, Spectrum for nonlinear maps and bifurcation in the non-differentiable case, in press.
5. M. FURI AND A. VIGNOLI, A nonlinear approach to surjectivity in Banach spaces *J. Functional Analysis* **20** (1975), 304–318.
6. P. R. HALMOS, "A Hilbert Space Problem Book," Van Nostrand, Princeton, N. J., 1967.
7. K. KURATOWSKY, "Introduction to Set Theory and Topology," 2nd ed., Pergamon Press, London/New York, 1972.
8. G. LUMER, Semi-inner product spaces, *Trans. Amer. Math. Soc.* **100** (1961), 29–43.
9. O. TOEPLITZ, Das algebraische Analogen zu einen Satze von Fejer, *Math. Zeit.* **2** (1918), 187–197.
10. K. YOSIDA, "Functional Analysis," Springer-Verlag, New York/Berlin, 1971.